Mathematica Moravica
Vol. **29**, No. **1** (2025), 31–41 doi: 10.5937/MatMor2501031M

On the Lichtenberg hybrid quaternions

Gamaliel Morales 0000-0003-3164-4434

Abstract. In this study, we define Lichtenberg hybrid quaternions. We give the Binet's formula, the generating functions, exponential generating functions and sum formulas of these quaternions. We find some relations between Jacobsthal hybrid quaternions, Mersenne hybrid quaternions and Lichtenberg hybrid quaternions. Also, Vajda's identity and, as consequences, Catalan's identity, d'Ocagne's identity and Cassini's identity are presented.

1. Introduction

Lichtenberg numbers are named after Georg Christoph Lichtenberg, who studied these numbers in the 17th century. Lichtenberg numbers are denoted by ℓ_n, defined mathematically by the recurrence $\ell_n + \ell_{n-1} = 2^n - 1$ and have the form

$$\ell_n = \frac{1}{6}\left[(-1)^{n+1} + 2^{n+2} - 3\right].$$

The first few terms of the Lichtenberg sequence are:

$$0, 1, 2, 5, 10, 21, 42, 85, 170, 341, 682, \dots \quad (A000975).$$

The Lichtenberg numbers $\{\ell_n\}_{n=0}^{\infty}$ are defined by the following recurrence relation

$$\ell_{n+3} = 2\ell_{n+2} + \ell_{n+1} - 2\ell_n, \tag{1}$$

with $\ell_0 = 0$, $\ell_1 = 1$ and $\ell_2 = 2$ (see, e.g. [7, 12]). Also, the Binet formula for Lichtenberg numbers is defined in two different ways, including well-known sequences of order 2:

$$\ell_n = \frac{1}{2}\left[\frac{2^{n+2} - (-1)^{n+2}}{3} - 1\right] = \frac{1}{2}\left[J_{n+2} - 1\right] \tag{2}$$

and

$$\ell_n = \frac{1}{3}\left[2^{n+1} - 1 - \frac{(-1)^n + 1}{2}\right] = \frac{1}{3}\left[M_{n+1} - \frac{(-1)^n + 1}{2}\right], \tag{3}$$

2020 *Mathematics Subject Classification.* Primary: 11B37; Secondary: 11B39, 20G20.
Key words and phrases. Hybrid numbers, Jacobsthal numbers, Jacobsthal quaternions, Lichtenberg numbers, Mersenne numbers, Mersenne quaternions.
Full paper. Received 14 Aug 2024, accepted 24 Feb 2025, available online 25 Feb 2025.

©2025 Mathematica Moravica

where J_n is the n-th Jacobsthal number (see [8]) and M_n is the n-th Mersenne number (see [1]).

The relation (3) can be rewritten as $\ell_n = \frac{1}{3}\left[M_{n+1} - b_n\right]$, where b_n is the usual sequence $b_n = \frac{(-1)^n+1}{2}$. The Jacobsthal numbers $\{J_n\}_{n=0}^{\infty}$ are defined by

$$J_{n+2} = J_{n+1} + 2J_n, \quad J_0 = 0, \quad J_1 = 1. \tag{4}$$

Also, the Mersenne numbers $\{M_n\}_{n=0}^{\infty}$ are defined by the following recurrence relation:

$$M_{n+2} = 3M_{n+1} - 2M_n, \quad M_0 = 0, \quad M_1 = 1. \tag{5}$$

Additionally, these exists the following relationships between Mersenne and Jacobsthal numbers (see, e.g. [1]):

$$M_n = \begin{cases} 3J_n, & \text{if } n \equiv 0 \pmod 2; \\ J_n + 4J_{n-1}, & \text{if } n \equiv 1 \pmod 2). \end{cases} \tag{6}$$

On the other hand, quaternions are generalizations of complex numbers and its multiplication is not commutative. Quaternions have generally the following form $\Psi = \psi_0 + \psi_1 i + \psi_2 j + \psi_3 k$, where $\{1, i, j, k\}$ are basis quaternions and ψ_m are real numbers for all $m = 0, 1, 2, 3$. The product rule for units of quaternions is defined by

$$i^2 = j^2 = k^2 = -1, \quad ijk = -1. \tag{7}$$

The author studied generalized Tribonacci quaternions numbers in their research in [2, 3] and applied it to numerous number sequences. Recently, Özdemir introduced the hybrid numbers and gave some of their properties (see [10]). The set of hybrid numbers is

$$\mathbb{R}[\mathbf{i}, \boldsymbol{\varepsilon}, \mathbf{h}] = \left\{\phi_0 + \phi_1\mathbf{i} + \phi_2\boldsymbol{\varepsilon} + \phi_3\mathbf{h} : \phi_r \in \mathbb{R},\ r = 0, 1, 2, 3\right\}.$$

The hybrid product is obtained by distributing the terms to the right, preserving the order of multiplication of the units and then writing the values of the following substituting each product of units by the equalities $\mathbf{i}^2 = -1$, $\boldsymbol{\varepsilon}^2 = 0$, $\mathbf{h}^2 = 1$ and $\mathbf{ih} = -\mathbf{hi} = \boldsymbol{\varepsilon} + \mathbf{i}$. The table 1 shows us that the multiplication operation with the hybrid numbers is not commutative. The author studied hybrid numbers in their work in [4] and applied it to many number sequences of order 2.

Table 1. The multiplication table for the basis of $\mathbb{R}[\mathbf{i}, \boldsymbol{\varepsilon}, \mathbf{h}]$.

$\times$	1	$\mathbf{i}$	$\boldsymbol{\varepsilon}$	$\mathbf{h}$
1	1	$\mathbf{i}$	$\boldsymbol{\varepsilon}$	$\mathbf{h}$
$\mathbf{i}$	$\mathbf{i}$	-1	$1-\mathbf{h}$	$\boldsymbol{\varepsilon}+\mathbf{i}$
$\boldsymbol{\varepsilon}$	$\boldsymbol{\varepsilon}$	$1+\mathbf{h}$	0	$-\boldsymbol{\varepsilon}$
$\mathbf{h}$	$\mathbf{h}$	$-(\boldsymbol{\varepsilon}+\mathbf{i})$	$\boldsymbol{\varepsilon}$	1

Using the Lichtenberg numbers $\{\ell_n\}_{n=0}^{\infty}$, we define Lichtenberg hybrid numbers $\{h\ell_n\}_{n=0}^{\infty}$ and Lichtenberg hybrid quaternions $\{Qh\ell_n\}_{n=0}^{\infty}$ and give some of their properties. We prove some theorems about Lichtenberg, Jacobsthal and Mersenne hybrid quaternions. We find generating functions, exponential generating functions, Binet formulas, sum formulas of these numbers and many other relationships between them.

2. Lichtenberg hybrid quaternion number $\{Qh\ell_n\}_{n=0}^{\infty}$

The Jacobsthal hybrid number $\{Jh_n\}_{n=0}^{\infty}$, is defined as

$$Jh_n = J_n + J_{n+1}\mathbf{i} + J_{n+2}\boldsymbol{\varepsilon} + J_{n+3}\mathbf{h}, \quad n \geq 0.$$

Similarly, the Mersenne hybrid number $\{Mh_n\}_{n=0}^{\infty}$, is defined as

$$Mh_n = M_n + M_{n+1}\mathbf{i} + M_{n+2}\boldsymbol{\varepsilon} + M_{n+3}\mathbf{h}, \quad n \geq 0.$$

The Binet formulas of the Jacobsthal and Mersenne hybrid numbers as follows:

$$Jh_n = \frac{1}{3}\left[2^n(1 + 2\mathbf{i} + 4\boldsymbol{\varepsilon} + 8\mathbf{h}) - (-1)^n(1 - \mathbf{i} + \boldsymbol{\varepsilon} - \mathbf{h})\right] \tag{8}$$

and

$$Mh_n = 2^n(1 + 2\mathbf{i} + 4\boldsymbol{\varepsilon} + 8\mathbf{h}) - (1 + \mathbf{i} + \boldsymbol{\varepsilon} + \mathbf{h}). \tag{9}$$

For more details on these two types of hybrid numbers see [13, 14].

Here, we introduce some properties Lichtenberg hybrid quaternions. We find some relations between Lichtenberg hybrid quaternion, Jacobsthal hybrid quaternion and Mersenne hybrid quaternion.

Definition 1. The Lichtenberg hybrid number is defined as follows:

$$h\ell_n = \ell_n + \ell_{n+1}\mathbf{i} + \ell_{n+2}\boldsymbol{\varepsilon} + \ell_{n+3}\mathbf{h}, \quad n \geq 0, \tag{10}$$

where $\{\mathbf{i}, \boldsymbol{\varepsilon}, \mathbf{h}\}$ are hybrid units, and ℓ_n is the n-th Lichtenberg number.

Definition 2. The Lichtenberg hybrid quaternion is defined as follows:

$$Qh\ell_n = h\ell_n + h\ell_{n+1}i + h\ell_{n+2}j + h\ell_{n+3}k, \quad n \geq 0, \tag{11}$$

where $\{i, j, k\}$ are quaternion units, and $h\ell_n$ is the n-th Lichtenberg hybrid number.

The Lichtenberg hybrid quaternion can be written as follows:

$$\begin{aligned}
Qh\ell_n &= h\ell_n + h\ell_{n+1}i + h\ell_{n+2}j + h\ell_{n+3}k \\
&= \ell_n + \ell_{n+1}\mathbf{i} + \ell_{n+2}\boldsymbol{\varepsilon} + \ell_{n+3}\mathbf{h} \\
&\quad + (\ell_{n+1} + \ell_{n+2}\mathbf{i} + \ell_{n+3}\boldsymbol{\varepsilon} + \ell_{n+4}\mathbf{h})\, i \\
&\quad + (\ell_{n+2} + \ell_{n+3}\mathbf{i} + \ell_{n+4}\boldsymbol{\varepsilon} + \ell_{n+5}\mathbf{h})\, j \\
&\quad + (\ell_{n+3} + \ell_{n+4}\mathbf{i} + \ell_{n+5}\boldsymbol{\varepsilon} + \ell_{n+6}\mathbf{h})\, k \\
&= Q\ell_n + Q\ell_{n+1}\mathbf{i} + Q\ell_{n+2}\boldsymbol{\varepsilon} + Q\ell_{n+3}\mathbf{h},
\end{aligned}$$

where $Q\ell_n = \ell_n + \ell_{n+1}i + \ell_{n+2}j + \ell_{n+3}k$ is the classic n-th Lichtenberg quaternion number.

Theorem 1. *For all $n \geq 0$, the Binet's formula of the Lichtenberg hybrid quaternion is given by:*

$$Qh\ell_n = \frac{1}{6}\left[(-1)^{n+1}\mathbf{a}_1\mathbf{b}_1 + 2^{n+2}\mathbf{a}_2\mathbf{b}_2 - 3\mathbf{a}_3\mathbf{b}_3\right], \tag{12}$$

where

$$\begin{aligned} \mathbf{a}_1 &= 1 - \boldsymbol{i} + \boldsymbol{\varepsilon} - \boldsymbol{h}, & \mathbf{b}_1 &= 1 - i + j - k,\\ \mathbf{a}_2 &= 1 + 2\boldsymbol{i} + 4\boldsymbol{\varepsilon} + 8\boldsymbol{h}, & \mathbf{b}_2 &= 1 + 2i + 4j + 8k,\\ \mathbf{a}_3 &= 1 + \boldsymbol{i} + \boldsymbol{\varepsilon} + \boldsymbol{h}, & \mathbf{b}_3 &= 1 + i + j + k. \end{aligned}$$

Proof. Using Binet formula of Lichtenberg number ℓ_n and (10), we have

$$\begin{aligned} h\ell_n &= \ell_n + \ell_{n+1}\mathbf{i} + \ell_{n+2}\boldsymbol{\varepsilon} + \ell_{n+3}\mathbf{h}\\ &= \frac{1}{6}\left[(-1)^{n+1} + 2^{n+2} - 3\right]\\ &\quad + \frac{1}{6}\left[(-1)^{n+2} + 2^{n+3} - 3\right]\mathbf{i}\\ &\quad + \frac{1}{6}\left[(-1)^{n+3} + 2^{n+4} - 3\right]\boldsymbol{\varepsilon}\\ &\quad + \frac{1}{6}\left[(-1)^{n+4} + 2^{n+5} - 3\right]\mathbf{h}\\ &= \frac{1}{6}\left[(-1)^{n+1}\mathbf{a}_1 + 2^{n+2}\mathbf{a}_2 - 3\mathbf{a}_3\right], \end{aligned}$$

where $\mathbf{a}_1 = 1 - \mathbf{i} + \boldsymbol{\varepsilon} - \mathbf{h}$, $\mathbf{a}_2 = 1 + 2\mathbf{i} + 4\boldsymbol{\varepsilon} + 8\mathbf{h}$ and $\mathbf{a}_3 = 1 + \mathbf{i} + \boldsymbol{\varepsilon} + \mathbf{h}$. Now, using Binet formula of Lichtenberg hybrid number $h\ell_n$ and (11), we have

$$\begin{aligned} Qh\ell_n &= h\ell_n + h\ell_{n+1}i + h\ell_{n+2}j + h\ell_{n+3}k\\ &= \frac{1}{6}\left[(-1)^{n+1}\mathbf{a}_1 + 2^{n+2}\mathbf{a}_2 - 3\mathbf{a}_3\right]\\ &\quad + \frac{1}{6}\left[(-1)^{n+2}\mathbf{a}_1 + 2^{n+3}\mathbf{a}_2 - 3\mathbf{a}_3\right]i\\ &\quad + \frac{1}{6}\left[(-1)^{n+3}\mathbf{a}_1 + 2^{n+4}\mathbf{a}_2 - 3\mathbf{a}_3\right]j\\ &\quad + \frac{1}{6}\left[(-1)^{n+4}\mathbf{a}_1 + 2^{n+5}\mathbf{a}_2 - 3\mathbf{a}_3\right]k\\ &= \frac{1}{6}\left[(-1)^{n+1}\mathbf{a}_1\mathbf{b}_1 + 2^{n+2}\mathbf{a}_2\mathbf{b}_2 - 3\mathbf{a}_3\mathbf{b}_3\right], \end{aligned}$$

where $\mathbf{b}_1 = 1 - i + j - k$, $\mathbf{b}_2 = 1 + 2i + 4j + 8k$ and $\mathbf{b}_3 = 1 + i + j + k$. Thus, the proof is completed. □

Using Theorem 1, we can deduce the following result.

Corollary 1. *For the n-th Lichtenberg hybrid quaternion, we have*

$$Qh\ell_n = \frac{1}{2}\big[QJh_{n+2} - \mathbf{a}_3\mathbf{b}_3\big], \tag{13}$$

where QJh_n is the n-th Jacobsthal hybrid quaternion.

Theorem 2. *The following equation is provided for Lichtenberg hybrid quaternion:*

$$Qh\ell_n + Qh\ell_{n-1} = 2^n \boldsymbol{a}_2\boldsymbol{b}_2 - \boldsymbol{a}_3\boldsymbol{b}_3. \tag{14}$$

Proof. Using (12), we have

$$\begin{aligned} Qh\ell_n + Qh\ell_{n-1} &= \frac{1}{6}\big[(-1)^{n+1}\mathbf{a}_1\mathbf{b}_1 + 2^{n+2}\mathbf{a}_2\mathbf{b}_2 - 3\mathbf{a}_3\mathbf{b}_3\big] \\ &\quad + \frac{1}{6}\big[(-1)^{n}\mathbf{a}_1\mathbf{b}_1 + 2^{n+1}\mathbf{a}_2\mathbf{b}_2 - 3\mathbf{a}_3\mathbf{b}_3\big] \\ &= \frac{1}{6}\big[3\cdot 2^{n+1}\mathbf{a}_2\mathbf{b}_2 - 6\mathbf{a}_3\mathbf{b}_3\big] \\ &= 2^n\mathbf{a}_2\mathbf{b}_2 - \mathbf{a}_3\mathbf{b}_3. \end{aligned}$$

Thus, the proof is completed. □

Remark 1. The sequence $2^n\mathbf{a}_2\mathbf{b}_2 - \mathbf{a}_3\mathbf{b}_3$ in Theorem 2 is recently studied by Özkan and Uysal in [11]. Furthermore, this sequence is called Mersenne hybrid quaternion and is denoted here as QMh_n. Other references on the hybrid quaternions are [9, 15].

Remark 2. A general sequence of order 2 is introduced by Dağdeviren and Kürüz (see [5]) considering hybrid and quaternion numbers. Also, the authors defined the Horadam hybrid quaternions and presented some of their properties. In fact, this new quaternion system is a combination of complex quaternions, hyperbolic quaternions and dual quaternions, and it can be viewed as a generalization of these quaternion systems (see [6]).

Theorem 3. *The recurrence relation of the Lichtenberg hybrid quaternion as follows:*

$$Qh\ell_{n+3} = 2Qh\ell_{n+2} + Qh\ell_{n+1} - 2Qh\ell_n, \quad n \geq 0. \tag{15}$$

Proof. We will use (1) and (11) for the proof. We have

$$\begin{aligned} Qh\ell_{n+3} &= h\ell_{n+3} + h\ell_{n+4}i + h\ell_{n+5}j + h\ell_{n+6}k \\ &= 2h\ell_{n+2} + h\ell_{n+1} - 2h\ell_n \\ &\quad + \big(2h\ell_{n+3} + h\ell_{n+2} - 2h\ell_{n+1}\big)i \\ &\quad + \big(2h\ell_{n+4} + h\ell_{n+3} - 2h\ell_{n+2}\big)j \\ &\quad + \big(2h\ell_{n+5} + h\ell_{n+4} - 2h\ell_{n+3}\big)k \\ &= 2\big(h\ell_{n+2} + h\ell_{n+3}i + h\ell_{n+4}j + h\ell_{n+5}k\big) \\ &\quad + h\ell_{n+1} + h\ell_{n+2}i + h\ell_{n+3}j + h\ell_{n+4}k \end{aligned}$$

$$\begin{aligned}&- 2\big(h\ell_n + h\ell_{n+1}i + h\ell_{n+2}j + h\ell_{n+3}k\big)\\ &= 2Qh\ell_{n+2} + Qh\ell_{n+1} - 2Qh\ell_n.\end{aligned}$$

Thus, the proof is completed. □

Theorem 4. *The generating function of the Lichtenberg hybrid quaternion is given by:*

(16)
$$\sum_{n=0}^{\infty} Qh\ell_n x^n = \frac{Qh\ell_0 + (Qh\ell_1 - 2Qh\ell_0)\,x + (Qh\ell_2 - 2Qh\ell_1 - Qh\ell_0)\,x^2}{1 - 2x - x^2 + 2x^3}.$$

Proof. Suppose that the generating function of the Lichtenberg hybrid quaternion sequence has the form $g(Qh\ell_n; x) = Qh\ell_0 + Qh\ell_1 x + Qh\ell_2 x^2 + \cdots + Qh\ell_n x^n + \cdots$. Now, multiplying (1) by x^{n+3} and then taking summation over 0 to ∞, we get

$$\begin{aligned}0 &= \sum_{n=0}^{\infty} x^{n+3}\big(Qh\ell_{n+3} - 2Qh\ell_{n+2} - Qh\ell_{n+1} + 2Qh\ell_n\big)\\ &= g(Qh\ell_n; x) - Qh\ell_0 - Qh\ell_1 x - Qh\ell_2 x^2\\ &\quad - 2\big[g(Qh\ell_n; x) - Qh\ell_0 + Qh\ell_1 x\big]x\\ &\quad - \big[g(Qh\ell_n; x) - Qh\ell_0\big]x^2\\ &\quad + 2\big[g(Qh\ell_n; x)\big]x^3\\ g(Qh\ell_n; x) &= \frac{Qh\ell_0 + (Qh\ell_1 - 2Qh\ell_0)\,x + (Qh\ell_2 - 2Qh\ell_1 - Qh\ell_0)\,x^2}{1 - 2x - x^2 + 2x^3}.\end{aligned}$$

Thus, the proof is completed. □

Theorem 5. *The exponential generating function of Lichtenberg hybrid quaternion has the form:*

$$\sum_{n=0}^{\infty} Qh\ell_n\Big(\frac{x^n}{n!}\Big) = \frac{1}{6}\big(-\mathbf{a}_1\mathbf{b}_1 + 4\mathbf{a}_2\mathbf{b}_2 e^{3x} - 3\mathbf{a}_3\mathbf{b}_3 e^{2x}\big)e^{-x}. \tag{17}$$

Proof. Using Binet formula of the Lichtenberg hybrid quaternion $Qh\ell_n$ in Theorem 1, we have

$$\begin{aligned}\sum_{n=0}^{\infty} Qh\ell_n\left(\frac{x^n}{n!}\right) &= \frac{1}{6}\sum_{n=0}^{\infty}\left[(-1)^{n+1}\mathbf{a}_1\mathbf{b}_1 + 2^{n+2}\mathbf{a}_2\mathbf{b}_2 - 3\mathbf{a}_3\mathbf{b}_3\right]\frac{x^n}{n!}\\ &= \frac{1}{6}\left[-\mathbf{a}_1\mathbf{b}_1\sum_{n=0}^{\infty}\frac{(-x)^n}{n!} + 4\mathbf{a}_2\mathbf{b}_2\sum_{n=0}^{\infty}\frac{(2x)^n}{n!} - 3\mathbf{a}_3\mathbf{b}_3\sum_{n=0}^{\infty}\frac{x^n}{n!}\right]\\ &= \frac{1}{6}\left[-\mathbf{a}_1\mathbf{b}_1 e^{-x} + 4\mathbf{a}_2\mathbf{b}_2 e^{2x} - 3\mathbf{a}_3\mathbf{b}_3 e^{x}\right]\\ &= \frac{1}{6}\left[-\mathbf{a}_1\mathbf{b}_1 + 4\mathbf{a}_2\mathbf{b}_2 e^{3x} - 3\mathbf{a}_3\mathbf{b}_3 e^{2x}\right]e^{-x}.\end{aligned}$$

Thus, the proof is completed. □

Theorem 6. *The sum of the Lichtenberg hybrid quaternion is given by:*

$$\sum_{r=0}^{n} Qh\ell_r = \frac{1}{2}\left[Qh\ell_{n+2} - (n+2)\mathbf{a}_3\mathbf{b}_3 - \mathbf{c}_1\mathbf{b}_3 - \mathbf{c}_2\right], \tag{18}$$

where $\mathbf{a}_3$, $\mathbf{b}_3$ *as in Theorem 1,* $\mathbf{c}_1 = \boldsymbol{i} + 4\boldsymbol{\varepsilon} + 9\boldsymbol{h}$, *and* $\mathbf{c}_2 = (1 + 3\boldsymbol{i} + 5\boldsymbol{\varepsilon} + 11\boldsymbol{h})i + (4 + 8\boldsymbol{i} + 16\boldsymbol{\varepsilon} + 32\boldsymbol{h})j + (9 + 19\boldsymbol{i} + 37\boldsymbol{\varepsilon} + 75\boldsymbol{h})k$.

Proof. Using (9), we have

$$\begin{aligned}\sum_{r=0}^{n} \ell_n &= \frac{1}{2}\left[\sum_{r=0}^{n+2} J_r - (n+2)\right]\\ &= \frac{1}{4}\big[J_{n+4} - (2n+5)\big]\\ &= \frac{1}{2}\big[\ell_{n+2} - (n+2)\big],\end{aligned}$$

where J_n is the n-th Jacobsthal number. Then, we can obtain the next result

$$\begin{aligned}\sum_{r=0}^{n} h\ell_r &= \sum_{r=0}^{n}\big[\ell_r + \ell_{r+1}\mathbf{i} + \ell_{r+2}\boldsymbol{\varepsilon} + \ell_{r+3}\mathbf{h}\big]\\ &= \sum_{r=0}^{n}\ell_r + \left(\sum_{r=0}^{n+1}\ell_r\right)\mathbf{i} + \left(\sum_{r=0}^{n+2}\ell_r - 1\right)\boldsymbol{\varepsilon} + \left(\sum_{r=0}^{n+3}\ell_r - 3\right)\mathbf{h}\\ &= \frac{1}{2}\big[h\ell_{n+2} - (n+2)\mathbf{a}_3 - \mathbf{c}_1\big],\end{aligned}$$

where $\mathbf{c}_1 = \mathbf{i} + 4\boldsymbol{\varepsilon} + 9\mathbf{h}$ and $\mathbf{a}_3$ as in Theorem 1.

Finally, using (11), we have

$$\begin{aligned}\sum_{r=0}^{n} Qh\ell_r &= \sum_{r=0}^{n}\big[h\ell_r + h\ell_{r+1}i + h\ell_{r+2}j + h\ell_{r+3}k\big]\\ &= \sum_{r=0}^{n} h\ell_r + \left(\sum_{r=0}^{n+1} h\ell_r - h\ell_0\right)i + \left(\sum_{r=0}^{n+2} h\ell_r - h\ell_0 - h\ell_1\right)j\\ &\quad + \left(\sum_{r=0}^{n+3} h\ell_r - h\ell_0 - h\ell_1 - h\ell_2\right)k\\ &= \frac{1}{2}\big[Qh\ell_{n+2} - (n+2)\mathbf{a}_3\mathbf{b}_3 - \mathbf{c}_1\mathbf{b}_3 - \mathbf{c}_2\big],\end{aligned}$$

where $\mathbf{c}_2 = (1+3\mathbf{i}+5\boldsymbol{\varepsilon}+11\mathbf{h})i + (4+8\mathbf{i}+16\boldsymbol{\varepsilon}+32\mathbf{h})j + (9+19\mathbf{i}+37\boldsymbol{\varepsilon}+75\mathbf{h})k$ and $\mathbf{b}_3$ as in Theorem 1. Thus, the proof is completed. □

Theorem 7. *For the n-th Lichtenberg hybrid quaternion, the following equations are provided:*

$$Qh\ell_{n+2} = Qh\ell_{n+1} + 2Qh\ell_n + \mathbf{a}_3\mathbf{b}_3 \tag{19}$$

and

$$Qh\ell_n = \frac{1}{6}\begin{cases} 2QMh_{n+1} - \mathbf{a}_1\mathbf{b}_1 - \mathbf{a}_3\mathbf{b}_3, & \text{if } n \equiv 0 \pmod 2; \\ 2QMh_{n+1} + \mathbf{a}_1\mathbf{b}_1 - \mathbf{a}_3\mathbf{b}_3, & \text{if } n \equiv 1 \pmod 2; \end{cases} \tag{20}$$

where QMh_n is the n-th Mersenne hybrid quaternion.

Proof. Using (12), we will prove the identity in (19). Then

$$\begin{aligned} Qh\ell_{n+1} + 2Qh\ell_n + \mathbf{a}_3\mathbf{b}_3 &= \frac{1}{6}\big[(-1)^{n+2}\mathbf{a}_1\mathbf{b}_1 + 2^{n+3}\mathbf{a}_2\mathbf{b}_2 - 3\mathbf{a}_3\mathbf{b}_3\big] \\ &\quad + \frac{1}{6}\big[2(-1)^{n+1}\mathbf{a}_1\mathbf{b}_1 + 2^{n+3}\mathbf{a}_2\mathbf{b}_2 - 6\mathbf{a}_3\mathbf{b}_3\big] + \mathbf{a}_3\mathbf{b}_3 \\ &= \frac{1}{6}\big[(-1)^{n+3}\mathbf{a}_1\mathbf{b}_1 + 2^{n+4}\mathbf{a}_2\mathbf{b}_2 - 3\mathbf{a}_3\mathbf{b}_3\big] \\ &= Qh\ell_{n+2}. \end{aligned}$$

Similarly, the proof of (20) can be done. □

For simplicity of notation, let

$$\mathcal{H}\ell_n = (-1)^{n+1}\mathbf{a}_1\mathbf{b}_1 + 2^{n+2}\mathbf{a}_2\mathbf{b}_2. \tag{21}$$

Then, the Binet formula of the Lichtenberg hybrid quaternions is given by

$$Qh\ell_n = \frac{1}{6}\big[\mathcal{H}\ell_n - 3\mathbf{a}_3\mathbf{b}_3\big], \tag{22}$$

where $\mathcal{H}\ell_0 = -\mathbf{a}_1\mathbf{b}_1 + 4\mathbf{a}_2\mathbf{b}_2$ and $\mathcal{H}\ell_1 = \mathbf{a}_1\mathbf{b}_1 + 8\mathbf{a}_2\mathbf{b}_2$.

The Vajda's identity for the sequence $\{\mathcal{H}\ell_n\}_{n\geq 0}$ and Lichtenberg hybrid quaternion sequence $\{Qh\ell_n\}_{n\geq 0}$ are given in the next theorem.

Theorem 8. *Let n, u and v be integers such that $n \geq 0$, $u \geq 0$ and $n+v \geq 0$. Then, we have*

$$\begin{aligned} &\mathcal{H}\ell_{n+u}\mathcal{H}\ell_{n+v} - \mathcal{H}\ell_n\mathcal{H}\ell_{n+u+v} \\ &= 3(-2)^{n+2}J_u\big[2^v\mathbf{a}_1\mathbf{b}_1\mathbf{a}_2\mathbf{b}_2 - (-1)^v\mathbf{a}_2\mathbf{b}_2\mathbf{a}_1\mathbf{b}_1\big], \end{aligned} \tag{23}$$

$$\begin{aligned} &Qh\ell_{n+u}Qh\ell_{n+v} - Qh\ell_nQh\ell_{n+u+v} \\ &= \frac{1}{12}\Big\{(-2)^{n+2}J_u\big[2^v\mathbf{a}_1\mathbf{b}_1\mathbf{a}_2\mathbf{b}_2 - (-1)^v\mathbf{a}_2\mathbf{b}_2\mathbf{a}_1\mathbf{b}_1\big] \\ &\qquad + \mathbf{a}_3\mathbf{b}_3\mathcal{K}\ell_n(u) - \mathcal{K}\ell_{n+v}(u)\mathbf{a}_3\mathbf{b}_3\Big\}, \end{aligned} \tag{24}$$

where $\mathcal{K}\ell_n(u) = \mathcal{H}\ell_n - \mathcal{H}\ell_{n+u}$.

Proof.
(23): Using (21), $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{b}_1$ and $\mathbf{b}_2$ as in Theorem 1, we have

$$\begin{aligned}
&\mathcal{H}\ell_{n+u}\mathcal{H}\ell_{n+v} - \mathcal{H}\ell_n\mathcal{H}\ell_{n+u+v} \\
&= \big((-1)^{n+u+1}\mathbf{a}_1\mathbf{b}_1 + 2^{n+u+2}\mathbf{a}_2\mathbf{b}_2\big)\big((-1)^{n+v+1}\mathbf{a}_1\mathbf{b}_1 + 2^{n+v+2}\mathbf{a}_2\mathbf{b}_2\big) \\
&\quad - \big((-1)^{n+1}\mathbf{a}_1\mathbf{b}_1 + 2^{n+2}\mathbf{a}_2\mathbf{b}_2\big)\big((-1)^{n+u+v+1}\mathbf{a}_1\mathbf{b}_1 + 2^{n+u+v+2}\mathbf{a}_2\mathbf{b}_2\big) \\
&= 3(-2)^{n+2}J_u\big[2^v\mathbf{a}_1\mathbf{b}_1\mathbf{a}_2\mathbf{b}_2 - (-1)^v\mathbf{a}_2\mathbf{b}_2\mathbf{a}_1\mathbf{b}_1\big],
\end{aligned}$$

where J_u is the u-th Jacobsthal number. Note that $\mathbf{a}_1\mathbf{b}_1\mathbf{a}_2\mathbf{b}_2 \neq \mathbf{a}_2\mathbf{b}_2\mathbf{a}_1\mathbf{b}_1$ in above equation.

(24): By formula (22) and (23), we get

$$\begin{aligned}
&36\big[Qh\ell_{n+u}Qh\ell_{n+v} - Qh\ell_nQh\ell_{n+u+v}\big] \\
&= \big[\mathcal{H}\ell_{n+u} - 3\mathbf{a}_3\mathbf{b}_3\big]\big[\mathcal{H}\ell_{n+v} - 3\mathbf{a}_3\mathbf{b}_3\big] - \big[\mathcal{H}\ell_n - 3\mathbf{a}_3\mathbf{b}_3\big]\big[\mathcal{H}\ell_{n+u+v} - 3\mathbf{a}_3\mathbf{b}_3\big] \\
&= \mathcal{H}\ell_{n+u}\mathcal{H}\ell_{n+v} - \mathcal{H}\ell_n\mathcal{H}\ell_{n+u+v} + 3\big[\mathbf{a}_3\mathbf{b}_3\mathcal{K}\ell_n(u) - \mathcal{K}\ell_{n+v}(u)\mathbf{a}_3\mathbf{b}_3\big] \\
&= 3(-2)^{n+2}J_u\big[2^v\mathbf{a}_1\mathbf{b}_1\mathbf{a}_2\mathbf{b}_2 - (-1)^v\mathbf{a}_2\mathbf{b}_2\mathbf{a}_1\mathbf{b}_1\big] \\
&\quad + 3\big[\mathbf{a}_3\mathbf{b}_3\mathcal{K}\ell_n(u) - \mathcal{K}\ell_{n+v}(u)\mathbf{a}_3\mathbf{b}_3\big],
\end{aligned}$$

where $\mathcal{K}\ell_n(u) = \mathcal{H}\ell_n - \mathcal{H}\ell_{n+u}$. □

It is easily seen that for special values of u and v by Theorem 8, we get new identities for Lichtenberg hybrid quaternions:

- Catalan's identity: $v = -u$ and $n \geq u$.
- Cassini's identity: $u = 1$, $v = -1$ and $n \geq 1$.
- d'Ocagne's identity: $u = 1$, $v = m - n$ and $m \geq n$.

Corollary 2. *Catalan identity for Lichtenberg hybrid quaternions. Let $n \geq 0$, $u \geq 0$ be integers such that $n \geq u$. Then*

$$\begin{aligned}
&Qh\ell_{n+u}Qh\ell_{n-u} - Qh\ell_n^2 \\
&= \frac{1}{12}\Big\{(-2)^{n+2}J_u\big[2^{-u}\mathbf{a}_1\mathbf{b}_1\mathbf{a}_2\mathbf{b}_2 - (-1)^{-u}\mathbf{a}_2\mathbf{b}_2\mathbf{a}_1\mathbf{b}_1\big] \\
&\qquad + \mathbf{a}_3\mathbf{b}_3\mathcal{K}\ell_n(u) - \mathcal{K}\ell_{n-u}(u)\mathbf{a}_3\mathbf{b}_3\Big\}.
\end{aligned}$$

Corollary 3. *Cassini identity for Lichtenberg hybrid quaternions. Let $n \geq 1$ be an integer. Then*

$$\begin{aligned}
&Qh\ell_{n+1}Qh\ell_{n-1} - Qh\ell_n^2 \\
&= \frac{1}{12}\Big\{(-2)^{n+1}\big[\mathbf{a}_1\mathbf{b}_1\mathbf{a}_2\mathbf{b}_2 + 2\mathbf{a}_2\mathbf{b}_2\mathbf{a}_1\mathbf{b}_1\big] \\
&\qquad + \mathbf{a}_3\mathbf{b}_3\mathcal{K}\ell_n(1) - \mathcal{K}\ell_{n-1}(1)\mathbf{a}_3\mathbf{b}_3\Big\}.
\end{aligned}$$

Corollary 4. *d'Ocagne identity for Lichtenberg hybrid quaternions. Let $n \geq 0$, $m \geq 0$ be integers such that $m \geq n$. Then*

$$\begin{aligned} &Qh\ell_{n+1}Qh\ell_m - Qh\ell_nQh\ell_{m+1} \\ &= \frac{1}{12}\Big\{(-2)^{n+2}\big[2^{m-n}\mathbf{a}_1\mathbf{b}_1\mathbf{a}_2\mathbf{b}_2 - (-1)^{m-n}\mathbf{a}_2\mathbf{b}_2\mathbf{a}_1\mathbf{b}_1\big] \\ &\qquad + \mathbf{a}_3\mathbf{b}_3\mathcal{K}\ell_n(1) - \mathcal{K}\ell_m(1)\mathbf{a}_3\mathbf{b}_3\Big\}. \end{aligned}$$

3. Conclusions and remarks

In this paper, we discussed the Lichtenberg hybrid quaternions and their properties. We obtained the Binet's formula, the generating function, exponential generating function and sum formula of these numbers. Further, we found some relations between Lichtenberg hybrid quaternion, Jacobsthal hybrid quaternion and Mersenne hybrid quaternion. In the future, the study of quadratic properties in this type of numbers could be encouraged, including the study of hybrid numbers with Lichtenberg hybrid number coefficients.

Acknowledgement

The author is thankful to the referee for valuable suggestions leading to improving the quality of the paper.

References

[1] P. Catarino, H. Campos and P. Vasco, *On the Mersenne sequence*, Annales Mathematicae et Informaticae, 46 (2016), 37–53.

[2] G. Cerda-Morales, *Identities for third order Jacobsthal quaternions*, Advances in Applied Clifford Algebras, 27 (2) (2017), 1043–1053.

[3] G. Cerda-Morales, *On a generalization of Tribonacci quaternions*, Mediterranean Journal of Mathematics, 14 (2017), Article ID: 239, 1–12.

[4] G. Cerda-Morales, *Investigation of generalized Fibonacci hybrid numbers and their properties*, Applied Mathematics E-Notes, 21 (2021), 110–118.

[5] A. Dağdeviren and F. Kürüz, *On the Horadam hybrid quaternions*, Advanced Studies: Euro-Tbilisi Mathematical Journal, 10 (2022), 293–306.

[6] A. Dağdeviren, *A generalization of complex, dual and hyperbolic quaternions: hybrid quaternions*, Filomat, 37 (25) (2023), 8441–8454.

[7] A.M. Hinz, *The Lichtenberg sequence*, Fibonacci Quarterly, 55 (2) (2017), 2–12.

[8] A.F. Horadam, *Jacobsthal representation numbers*, Fibonacci Quarterly, 43 (1) (1996), 40–54.

[9] M.C.S. Mangueira, F.R.V. Alves and P.M.M.C. Catarino, *Hybrid quaternions of Leonardo*, Trends in Computational and Applied Mathematics, 23 (1) (2022), 51–62.

[10] M. Özdemir, *Introduction to hybrid numbers*, Advances in Applied Clifford Algebras, 28 (2018), 1–32.

[11] E. Özkan and M. Uysal, *On the Mersenne and Mersenne-Lucas hybrid quaternions*, Bulletin of the Transilvania University of Brasov, Series III: Mathematics and Computer Science, 4 (1) (2024), 139–148.

[12] P.K. Stockmeyer, *An exploration of sequence A000975*, Fibonacci Quarterly, 55 (5) (2017), 174–185.

[13] A. Szynal-Liana and I. Włoch, *On Jacobsthal and Jacobsthal-Lucas hybrid numbers*, Annales Mathematicae Silesianae, 33 (2019), 276–283.

[14] D. Taşci and E. Sevgi, *Some properties between Mersenne, Jacobsthal and Jacobsthal-Lucas hybrid numbers*, Chaos, Solitons & Fractals, 146 (2021), Article ID: 110862, 1–7.

[15] M. Uysal and E. Özkan, *Padovan hybrid quaternions and some properties*, Journal of Science and Arts, 22 (1) (2022), 121–132.

Gamaliel Morales
Instituto de Matemáticas
Pontificia Universidad Católica de Valparaíso
Blanco Viel 596, Valparaíso
Chile
E-mail address: gamaliel.cerda.m@mail.pucv.cl